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Orientation of a Triangle

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Commentary

Abstract

The sphere, torus, Klein bottle, and the projective plane are the classical examples of orientable and non-orientable surfaces. In this paper, we give a general review on orientation of a triangle and some applications to topological objects.

Keywords— Triangulation, Orientation triangle, Simplex, Manifold, Homeomorphism, Polyhedron

1 Introduction

Alongside the studyof analysis and algebra, topology is considered as one of the areas of pure mathematics [Obeng-Denteh,]. It is from the idea of geometry (solid plane) by loosing the structures to different types of shapes, by preserving it structure [Obeng-Denteh and Manu, 2012]. Its applications is seen in various areas of mathematics.

Homology is an aspect of algebraic topology that looks at differentiating between spaces by constructing algebraic invariants that relates the connectivity properties of the space. In other words it is the way of connecting sequence of algebraic objects (such as modules or abelian groups) to other mathematical objects such as the topological space, it is mostly defined in Algebraic Topology[Obeng Denteh, 2019, Eilenberg and Cartan, 1956].

The most commonly known triangulated categories arise from chain complexes in an abelian category by passing to chain homotopy classes or inverting quasi-isomorphisms. Such examples are called 'algebraic' because they originate from abelian (or at least additive) categories. Stable homotopy theory produces examples of triangulated categories by quite different means, and in this context the source categories are usually very 'non-additive' before passing to homotopy classes of morphisms [Schwede, 2008].

One of the powerful tools in studying the properties of a manifold is by triangulating it to construct what we call homeomorphism to a simplicial complex [Manolescu, 2014].

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2 Preliminaries

2.1 Classification of Surfaces

A topological space X in which each point has a neighbourhood homeomorphism to an open disc we all a two-dimensional manifold. It is therefore convenient to study these surfaces by dividing them into pieces which are topologically equivalent to triangles on the two-dimensional Euclidean plane.

Definition 2.1 (Orientation of a manifold [FRIEDL, 2014])**.** *Let V be a vector space of dimension $k \geq 1$.*

(1) The determinant of the base change of a matrix is positive if the bases of two vectors are equivalent.

(2) An equivalence class of bases is called an orientation for V. An oriented vector space includes the vector space itself with its orientation.

(3) Let (V,O) be an oriented vector space. Say $\{v_1, \cdots, v_k\}$ is a positive basis, if $\{v_1, \cdots, v_k\}$ lies in O, otherwise we say it is a negative basis.

Definition 2.2. *Call the pair (τ, ψ) where τ is a subspace in X and taking $\psi : \triangle \to \tau$ is a homeomorphism of some triangle $\triangle \subset \mathbb{R}^2$ and τ, a topological triangle in X.*

- If $\psi : \triangle \to \tau$ is fixed, then $\tau \subset X$ is called a topological triangle.
- Images of the vertices are called vertices and sides of triangle $\triangle$ are called edges.

Definition 2.3 (Simplicial Complex)**.** *1. Given k points $p_1, \cdots, p_k$ in $\mathbb{R}^n$ such that $p_1, \cdots, p_k$ as a k-simplex.*

2. Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \cdots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e.

$$\triangle^n := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | x_0 + \cdots + x_n = 1\}$$

and

$$x_i \geq 0 \, \forall i = 0, \cdots, n$$

3. Given the standard n-simplex $\triangle^n$, we define its boundary as

$$\partial \triangle^n = \bigcup_{t=0}^{n} \{(t_0, \cdots, t_n) \in \triangle^n | t_i = 0\}$$

Definition 2.4. *[Manolescu, 2014] A piecewise linear manifold is a topological manifold equipped with a (maximal) open cover by charts such that the transition maps are piecewise linear.*

3 Main Thrust

3.1 Orientation of a triangle

The orientation of triangles arise from chain complexes in an abelian category. Given the vertices of $\triangle$, one can form different ordered triples of points. We therefore say that, two triples are equivalent if they can be formed from each other by a cyclic permutation, given us a two equivalence classes. Thus a $\triangle$ is oriented if one of the equivalence class is given. If the triangle $\triangle$ is oriented, then a topological triangle (τ, ϕ) is also oriented. The orientation determines the direction of going around its vertices, and these vertices determines, through the homeomorphism ϕ, a direction of going around the vertices of the topological triangle, which also determines orientations of its edges [Borisovich et al., 2013]. An oriented manifold is a manifold with its orientation, if the a manifold admits an orientation, then it is orientable, otherwise called non-orientable [FRIEDL, 2014].

Theorem 1. *[Matumoto, 1978] Let $n \geq 6$. Then, all the topological n-manifolds without boundary are simplicially triangulable if and only if there exists a smooth homology 3-sphere H^3 which satisfies the following conditions:*

- *H^3 bounds a parallelizable smooth 4-manifold with signature 8,*
- *the $(n-3)$-ple suspension of H^3 is homeomorphic to the n-sphere, and*
- *the oriented connected sum $H^3 \times H^3$ bounds an acyclic smooth 4-manifold.*

Definition 3.1. *[Borisovich et al., 2013] A finite set $K = \{(\tau_i, \psi_i)\}_{i=1}^{K}$ of topological triangles in X that fulfills the conditions*

(1) $X = \bigcup_{i=1}^{K} \tau_i.$

(2) the intersection of any pair of triangles from K is either empty or coincides with their common vertex or common edge is called a triangulation of the 2-D manifold of X.

A manifold which has a triangulation is said to be triangulable.

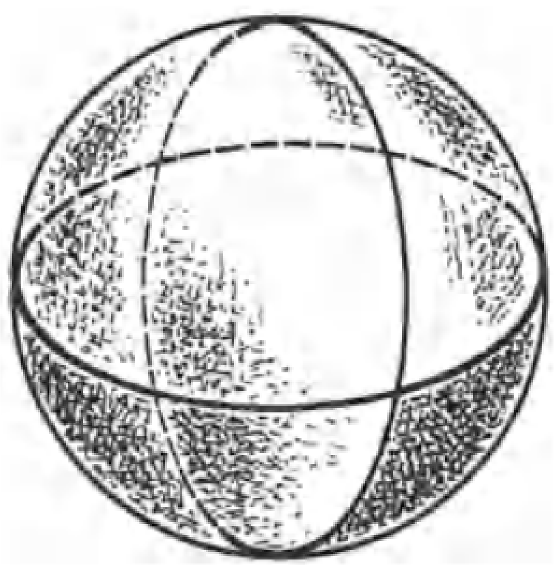

Figure 1: Triangulation of the sphere S^2 consisting of eight triangles [Borisovich et al., 2013].

3.2 Triangulation

One of the fundamental basics of building blocks or chains in topology for spaces is the k-simplex. A 0-simplex is considered a point, a 1-simplex also a closed interval, a 2-simplex is a triangle and a 3-

simplex is a tetrahedron. Generally, the k-simplex is the convex hull of $k+1$ vertices in k-dimensional space.

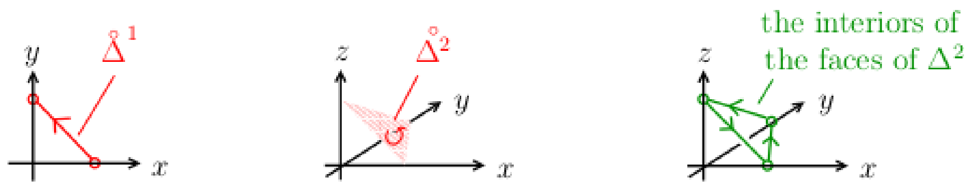

Figure 2: Standard simplices in dimensions 0,1 and 2 [FRIEDL, 2014].

The beauty of this triangulation can be viewed in $\triangle^1 \times [0,1]$ which is the union of triangles with vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.

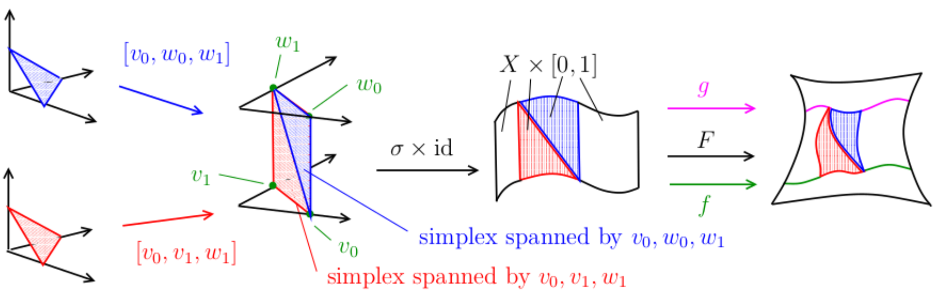

Figure 3: Standard simplices in dimensions 0,1 and 2 [FRIEDL, 2014].

Constructing of complicated spaces can now be done by sticking together many of these simplices along their faces. and a space constructed in this fashion is called a simplicial complex [Manolescu, 2014]. An example can be seen with the cube which can be formed out of twelve triangles-two for each face as seen below.

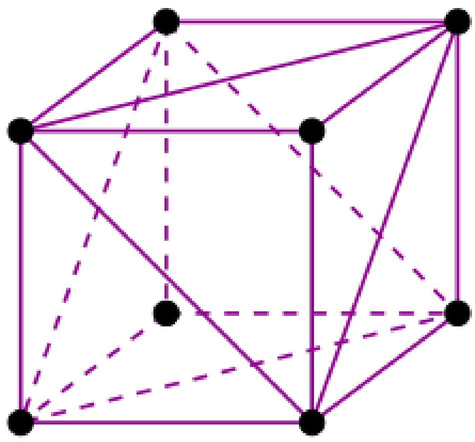

Figure 4: A cube [Manolescu, 2014].

A similar example is with that of a polygon, a closed surface. With any number of sides, one can cut along the diagonals into smaller triangles, this also implies that with the edges of any closed surfaced identified, triangles can be constructed. The triangulation is one of the important tools from which a single block is constructed into topological triangles, their vertices and surfaces.

Definition 3.2. *We call a connected, triangulable, two-dimensional manifold a closed surface.*

Definition 3.3. *A space $|K|$ and, more generally, any topological space X which is homeomorphic to K, is called a polyhedron.*

Definition 3.4. *A triangulation of a polyhedron X is a simplicial complex K such that the space K is homeomorphic to X.*

The study of a manifold does not in it entirety tell us much about the global structure. In constructing a heomomorphism to a simplicial complex of the properties of a manifold, it is then useful to triangulate it. An example is seen in that of the sphere which is homeomorphic to the cube, thus, it also has a triangulation with twelve triangles. The result of a triangulation gives combinatorial description of a manifold. To tell if two manifolds are apart, then one needs to check if their topological invariants are same (that is, homology groups). If one is able to triangulate the manifolds, then we can find their homology groups in terms of these triangulations. Poincare [Poincaré, 1900] was the first to formulate a question on triangulating manifolds. It reads today as

Question 3.1. *Does every smooth manifold admit a triangulation?*

Other researchers in 1926 asked this same question in a more general way.

Question 3.2. *Does every topological manifold admit a triangulation? [Manolescu, 2014]*

4 Application of triangulation

Definition 4.1. *[Teo, 2011] A polyhedron is a topological space that is homeomorphic to an Euclidean simplicial complex*

Definition 4.2. *[Teo, 2011] A triangulation is a particular homeomorphism between a topological space and a Euclidean simplicial complex.*

Theorem 2 (Triangulation theorem for 2-manifolds)**.** *[Teo, 2011] Every 2-Manifold is homeomorphic to the polyhedron of a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.*

Cutting a square along a diagonal produces two triangles, so each of these surfaces can also be built from two triangles by identifying their edges in pairs. Using only triangles we could also construct a large class of 2 dimensional spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time.

The piecewise linear manifolds are also easy to triangulate because it admits a combinatorial triangulation for which the structure of manifold is very evident. the manifold structure is evident.

The torus, Klein bottle, and the projective plane are the classical examples of orientable surfaces with triangulation. From $I \times I \sim$ with the equivalence relation given by $(x, 0) \sim (x, 1) \forall x \in I$ and

$(0, y) \sim (1, y) \forall y \in I$ is homeomorphic to a torus. We can make $I \times I$ into a simplicial complex k as seen below;

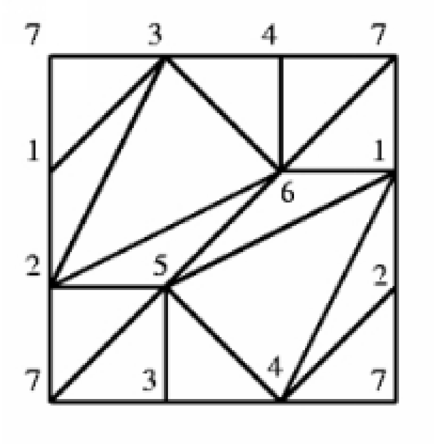

Figure 5: The minimal triangulation of the torus [Teo, 2011].

5 Real life application of orientation of a triangle

The applications is seen in the study of various aspect of mathematics. The applications is not limited to;

1. the geometry of a football is formed by triangulation (patches).
2. the formation of team players in a football game is that of a triangulated formation. A 4-3-3 formation is to get every player to be among one positional triangle in relation to his teammates.
3. trusses consist of triangle-shaped elements only used for roofing because a triangle is rigid and cannot be distorted.
4. experts in medical fields have suggested that flexible endoscopic surgery emulate the principles of triangulation which is seen to be an efficient surgical approach [Spaun et al., 2009].

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