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General Overview of Cell Complexes

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Commentary

Abstract

A very large class of topological spaces of practical interest can be represented by a decomposition into subsets, each with simple topology, glued together along their boundaries. A decomposition of this form is commonly called a cell complex. Abstract graphs (as branched 1-manifolds) are examples of cell complexes, as are quadrilateral and hexahedral meshes, and 3d triangulations supporting normal surfaces. There are several different ways to formalize the intuitive notion of a 'cell complex', striking different balances between simplicity and generality. I'll describe different definitions and how they are interconnected.

Keywords: Topological space, Cell complex, Homotopy, Simplicial Complex, *CW* Complex, Homology, Subcomplex, Simplex, Cells.

1 Introduction

A cell complex is a type of topological space to meet the needs of homotopy theorem. A cell complex as a class of spaces is broader and has some better categorical properties than simplicial complexes but still remains a combinatorial nature that allows for computations. It can be constructed from simplexes of different dimensions [1].

1.1 Types of Cell Complexes

1. Simplicial Complexes
2. CW Complexes

2 Notations and Preliminaries

Notations:

- D^n is an n-dimensional ball
- S^{n-1} is an (n-1)dimensional sphere.
- $A \subset X$ means A is a subset of X.
- $A \in B$ means A is in B.
- $\partial D^n = S^{n-1}$: the boundary of the n disk.
- e^n : an n cell, homeomorphic to the open n disk $D^n - \partial D^n$.

Preliminaries

Definition 1

A topology on a set X is a collection τ of open subsets of X satisfying:

1. Any union of elements of τ belongs to τ.
2. Any finite intersection of elements of τ belongs to τ.
3. ϕ and X belongs to τ.

We then say, (X, τ) is a topological space [2].

Definition 2

A simplicial complex $K = (E, S)$ consists of a set E of vertices and a set S of finite non-empty subsets of E. A set $s \in S$ with $q + 1$ elements is called a q-simplex of K. We require the following axioms:

1. $\{e\} \in S$ for each $e \in E$.
2. If $t \in S$ and $s \subset t$ is non-empty, then $s \in S$.

[3]

Definition 3

Let X be a cell complex. $A \subset X$ is a subcomplex of X if A is a union of cells of X such that the closure of each cell in A is contained in A

Definition 4

A Cell or an i-cell is a space of the following form:
$e^i = \{x \in \mathrm{R}^i | \; ||x|| < 1\}, \forall i = 0, 1, 2, ...$ The integer i stands for the dimension of the cell.

Definition 5

Let X, Y be topological spaces, and $f, g : X \mapsto Y$ continuous maps.
A homotopy from f to g is a continuous function $F : X \times [0, 1] \mapsto Y$ satisfying $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. If such a homotopy exists, we say that f is homotopic to g, and denote this by $f \simeq g$ [4].

Definition 6

Let $f : X \mapsto Y$ be a continuous map. Then f is said to be homotopy equivalent if there exists a continuous map $g : Y \mapsto X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ [4].

Definition 7

Identification Topology. Let X be a topological space and let Y be an arbitrary set and let $p : X \mapsto Y$ be a surjection. Then we can define a topology in$Y$ by: a subset $U \subseteq Y$ is open iff $p^{-1}(U)$ is open in X. This topology is the largest topology in Y for which $p : X \mapsto Y$ is continuous. We call it the identification topology in Y determined by p, and $p : X \mapsto Y$ is called an identification map.
If X and Y are two spaces and $p : X \mapsto Y$ a surjective map, then p is called an identification map if the topology in Y is the identification topology determined by p.

2.1 Definition 8

Abstract simplicial complex: A collection S of finite non-empty sets such that all the non-empty subsets of an element of S also belong to S; if $A \in S$ then for all $B \in 2^A$ with $B \neq \phi$, $B \in S$; every abstract simplicial complex has a unique representation as a simplicial complex, up to a linear isomorphism.

3 Main Thrust

3.1 Definition

A CW complex is any topological space X built in the following way:

1. You start with the empty set, and attach a collection of 0-cells (points: the "boundary of a point" is the empty set, so the attaching map is the unique map from the empty set to the empty set). The result is a discrete space (just a bunch of points) called X^0 (the 0-skeleton of X^1).

2. You add 1-cells e (possibly infinitely many) by specifying attaching maps $\partial e \mapsto X^0$. The result is called the 1-skeleton X^1.

3. You add 2-cells e (possibly infinitely many) by specifying attaching maps $\partial e \mapsto X^1$. The result is called the 2-skeleton X^2. You continue in this manner, constructing a nested sequence of skeleta $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset$

4. You take the union $X = \bigcup_{n \geq 0} X^n$ of all skeleta and equip it with the weak topology, in which a subset $U \subset X$ is open if and only if $U \cap X^n$ is open for all $n \geq 0$ [5].

3.2 Proposition 1

A compact subspace of a CW complex is contained in a finite subcomplex.
Proof:
First we show that a compact set C in a CW complex X can meet only finitely many cells of X . Suppose on the contrary that there is an infinite sequence of points $x_i \in C$ all lying in distinct cells. Then the set $S = \{x_1, x_2, \dots\}$ is closed in X . Namely, assuming $S \cap X^{n-1}$ is closed in X_{n-1} by induction on n, then for each cell e_α^n of X , ρ_α^{-1}(S) is closed in ∂D_α^n and ϕ_α^{-1}(S) consists of at most one more point in D_α^n, so ϕ_α^{-1}(S) is closed in D_α^n Therefore $S \cap X^n$ is closed in X^n for each n, hence S is closed in X .
The same argument shows that any subset of S is closed, so S has the discrete topology. But it is compact, being a closed subset of the compact set C . Therefore S must be finite, a contradiction. Since C is contained in a finite union of cells, it suffices to show that a finite union of cells is contained in a finite subcomplex of X . A finite union of finite subcomplexes is again a finite subcomplex, so this reduces to showing that a single cell e_α^n is contained in a finite subcomplex. The image of the attaching map ρ_α for e_α^n is compact, hence by induction on dimension this image is contained in a finite subcomplex $A \subset X^{n-1}$. So e_α^n is contained in the finite subcomplex $A \cup e_\alpha^n$ [5].

4 Concluding Remarks

In as much as cell complexes may differ in diverse ways, it still has the same concept and can be applied to any of its types.

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