Volterra Integral Equations: Candid Appreciation

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Abstract
Volterra integral equations have many applications in various disciplines of sciences that has attracted the attention of several mathematicians and scientists who work have worked in the areas in which these equations appear. Hence, it is worth to study these equations from its theoretical point of view due to its significance and broadly usage of these equations in numerous fields of mathematics. In this research study, the first and the second kind of Volterra integral equations have been defined. The research study summarizes some of the basic concepts and results of these equations and are discussed theoretically.

Keywords: Volterra Integral Equations, First Type, Second Type, Linear, Nonlinear.

Introduction
According to Wazwaz (2011), Integral equations occur naturally in many fields of science and engineering and has many applications (Obeng-Denteh, 2015). Scientists and mathematicians encounter integral equations in their various scientific discipline and it helps them to understand the natural phenomena effectively (Poltnikov & Skripnik, 2013). Among the various fields in which various forms of Integral equations are encountered include: continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radiative equilibrium, the particle transport
problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, and radiative heat transfer problems (Polyanin & Manzhirov, 2008).

History has it that, Joseph Fourier set out to initiate the theory of integral equations when he worked on the well – known *Fourier Transforms and its inversion* formula (Corduneanu, 1991). Apparently, another French mathematician known as Joseph Liouville also worked on integral equations in the 1840’s and discovered that, certain types of second order linear ordinary differential equation under some initial conditions is equivalent to an integral equation known later as a Volterra Integral Equation of the second type (Corduneanu, 1991).

However, in 1895 a new era for the theory of integral equations started under the term of Volterra, not like other mathematicians who tried to formulize the solutions of integral equations and worked on special cases. Volterra studied these equations from a functional analytic point of view and was interested in the existence of the solutions of these equations. He was also interested in the applications of these equations and one of the interesting applications is *hereditary mechanics* which was observed when Volterra was inspecting a population growth model (Rahman, 2007).

Volterra integral equations has many applications and literature has it that many scientists and mathematician have used it in many aspects of their endeavours (Issaka, 2016; Wazwaz, 2011). In this research study, the authors summarize some of the basic concepts and results of these equations and are discussed theoretically.

**Basic Definitions**

**Definition 1.** An integral equation is an equation that involves the unknown function \( u(x) \) that appears inside of an integral sign. The most standard type of an integral equation in \( u(x) \) is of the form:

\[
 u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt \ldots (1)
\]
Definition 2. If the exponent of the unknown function \( u(x) \) inside the integral sign in (1) is one, the integral equation is called linear. If the unknown function \( u(x) \) has exponent other than one, or if the equation contains nonlinear functions of \( u(x) \), the integral equation is called nonlinear.

Definition 3. If the function \( f(x) = 0 \) in equation (1), then equation (1) becomes homogeneous, otherwise it is called non-homogeneous,

\[
i.e.\ u(x) = \lambda \int_{0}^{x} k(x,t)u(t)dt \ldots \ldots (2)
\]

Definition 4. Equation (1) is called singular if one of the limits of integration \( g(x) \), \( h(x) \), or both are infinite, or if the kernel \( K(x, t) \) becomes unbounded at one or more points in the interval of integration:

\[
u(x) = \lambda \int_{g(x)}^{x} k(x,t)u(t)dt \ldots \ldots (3)
\]

\[
u(x) = \lambda \int_{\infty}^{x} k(x,t)u(t)dt \ldots \ldots (4)
\]

Definition 5. If at least one limit of the integral in equation (1) is a variable, then equation (1) is called a Volterra integral equation. These types of equations are classified into two types (First type and Second type).

The general form of Volterra integral equation of the first type is given by:

\[
f(x) = \lambda \int_{0}^{x} K(x,t)u(t)dt \ldots \ldots (5)
\]

where the unknown function \( u(x) \) appears inside the integral sign.

The general form of the Volterra integral equations of the second type is given by:

\[
u(x) = f(x) + \lambda \int_{0}^{x} K(x,t)u(t)dt \ldots \ldots (6)
\]

where the unknown function \( u(x) \) appears inside and outside the integral sign.
Note: $K(x, t)$ is a known continuous function in two variables and it is referred to as the Kernel of the Integral Equation. $f(x)$ is the free or forcing term which is also known and continuous. $u(x)$ is the unknown function, $a$ and $b$ are constants and the limits of integration and $\lambda$ is a numeric parameter. The Kernel of the equation $K(x, t)$ is defined on the $x \rightarrow t$ in the square $R$, where $R: a \leq x \leq b; a \leq t \leq b$. The Kernel $K(x, t)$ and $f(x)$ are continuously differentiable functions and $K(x, t) \neq 0$ for $a \leq x \leq b$.

Conclusion
The most significant types of mathematical equations that occur in natural science are differential equations and integral equations, this is because they help in the understanding of their natural phenomena and in modeling them mathematically.

Apparently, numerous significant phenomena in dynamical systems of economics, biology and electrical engineering can be modeled by Volterra Integral Equations. The theory of Volterra Integral Equations has attracted the attentions of many scientists. These scientists worked in terms of the development of this theory and its applications in numerous disciplines.

The basic theorems and the definition of the various forms of the Volterra Integral Equations were discussed in the research study. It is the hope of the researchers that this research study will enlighten students and other researchers in this field of Volterra Integral Equations and its applications as well.

References
