Calculation of Homology Group of Simplicial Complexes

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Abstract
Dwelling on the geometric interpretation of the homology groups of a simplicial complexes the thrust is the geometric meaning of the connecting homomorphism. I will briefly talk about the geometric interpretation of the homology groups of a simplicial complexes before talking about the main thrust. Let K be a simplicial complex and also let L be a subcomplex of a simplicial complex K. Then we say that $C_\ast (L; G)$ is a subcomplex of the chain complex $C_\ast (K; G)$. Hence the quotient complex is defined $C_\ast (K, L; G) = C_\ast (K; G) / C_\ast (L; G)$.

Keywords:
Homology class, connecting homomorphism, relative homology group, relative cycle, simplicial complexes, boundary homomorphism, oriented simplexes.

Preliminaries:
The homology group of this chain complex $C_\ast (K, L; G)$ will be denoted by $H_k (K, L; G)$. We can obtain a long exact sequence of homology group from the exact sequence of chain complexes, that is, $0 \rightarrow C_\ast (L; G) \xrightarrow{i_\ast} C_\ast (K; G) \xrightarrow{j_\ast} C_\ast (K, L; G) \rightarrow 0$ which produces $\ldots \rightarrow H_{k+1} (K, L; G) \xrightarrow{\delta_{k+1}} H_k (L; G) \xrightarrow{i_{*k}} H_k (K; G) \xrightarrow{j_{*k}} H_k (K, L; G) \xrightarrow{\delta_k} H_{k-1} (L; G) \rightarrow \ldots$ and this is called an exact sequence of the pair $(K, L)$ and also the group $H_k (K, L; G)$ are called relative homology groups or homology groups of the pair $(K, L)$. Let the chain $\tilde{Y}_k$ from $C_k (K, L; G)$ be a coset of
the group $C_k(K; G)$ relative to the subgroup $i_kC_k(L; G) \simeq C_k(L; G)$. We can show that in the coset $\bar{Y}_k$, there exists a unique representative, the chain $*$ from $C_k(K; G)$ which includes only these oriented simplexes with nonzero coefficients of the complex $K$ that are not oriented simplexes of the subcomplex $L$. It follows that the boundary homomorphism $\delta_k : C_k(K, L; G) \rightarrow C_{k-1}(K, L; G)$ transforms the chain $\bar{Y}_k$ into a chain $\bar{Y}_{k-1}$ which is the coset of the group $C_{k-1}(K; G)$ relative to the subgroup $i_{k-1}C_{k-1}(L; G) \simeq C_{k-1}(L; G)$ with the representative $\delta_kY_k \in C_{k-1}(K; G)$.

**Main Thrust**

Now the main thrust which talks about the geometric meaning of the connecting homomorphism $\delta_k : H_k(K, L; G) \rightarrow H_{k-1}(L; G)$. Let $h_k \in H_k(K, L; G)$ be a homology class of the relative cycle $Z_k \in \tilde{C}_k$. We want to show that $Z_k$ is a chain in $C_*(K; G)$ and calculate its boundary $\delta_kZ_k$ in it. We know from relative cycle that the chain in $\delta_kZ_k$ will include nonzero coefficient only oriented simplexes from $L$. Therefore $\delta_kZ_k$ is a chain $C_*(L; G)$. Hence $\delta_kZ_k$ is a cycle whose homology class $h_{k-1} \in H_{k-1}(L; G)$ does not depend on the choice of the representative $\delta_k$ of the class $\tilde{h}_k$. Therefore the general structure of the connecting homomorphism $\delta_k\tilde{h}_k=h_{k-1}$. Now let's take an example. Let $U, V, W, X$ be simplexes of a rectangle of a simplicial complex $K$.

Thus $(U, V), (V, W), (W, X), (X, U), (V, X), (U, V, X), (V, W, X)$. And let its subcomplex $L$ consist of the sample simplexes except $(V, W), (U, V, X), (V, W, X)$. Thus $|K|$ is a rectangle (with the interior) and $|L|$ its boundary. So from this, the chain $Y_2 \in C_2(K, Z), Y_2 = [U, V, X] + [V, W, X]$ is a relative cycle of a pair $(K, L)$. The boundary $\delta_2Y_2 = [U, V] + [V, W] + [W, X] + [X, U]$ includes with nonzero coefficient only oriented simplexes from $L$. Also the chain $Y_1 [V,
X] from $C_1(K; Z)$ is simultaneously a relative cycle and a relative boundary because it can be obtained from $\Upsilon_2 [U, V, X] = [V, X] + [X, U] + [U, V]$. By discarding the addends $[X, U]$ and $[U, V]$ which are oriented simplexes from the subcomplex $L$. Then it proves that the relative cycle $\Upsilon_2$ determines the generator of the group $H_2(K, L; Z) \cong Z$. Hence the connecting homomorphism $\delta_2: H_2(K, L; Z) \to H_1(L; Z)$ associates this generator with an element of the group $H_1(L; Z)$ which consists of one cycle $\delta_2 \Upsilon_2$.

**Concluding Remarks**

I will conclude by saying that from the above example, it proves that a simplicial complex $K$ can be formed from a relative cycle and a relative boundary of a subcomplex $L$. So by discarding the addends of the oriented simplexes from the subcomplex $L$, the relative cycle determines the generator of the homology group.

**Reference**

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